Introduction: Paraconsistent Logics

The papers in this volume are all on the subject of paraconsistency. This introduction locates the papers in their context and also provides a survey of the general area.

1. Paraconsistency: its definition and its rationale

Let $\Vdash$ be a relation of logical consequence. $\Vdash$ may be defined either semantically ($\Sigma \Vdash A$ holds iff for some specified set of valuations, wherever all the formulas in Σ are true under an evaluation, so is A) or proof theoretically ($\Sigma \Vdash A$ holds iff for some specified set of rules, there is a derivation of A, all of whose (undischarged) premises are in Σ), or in some other way. $\Vdash$ is *explosive* iff for all A and B $\{A, \sim A\} \Vdash B$. It is *paraconsistent* iff it is not explosive. A logic is *paraconsistent* iff its logical consequence relation is[0].

Let Σ be a set of sentences. Σ is *inconsistent* iff, for some A, $\{A, \sim A\} \subseteq \Sigma$. Σ is *trivial* iff for all B, $B \in \Sigma$. The important fact about paraconsistent logics is that they provide the basis for inconsistent but non-trivial theories. In other words, there are sets of sentences closed under logical consequence which are inconsistent but non-trivial. This fact is sometimes taken as an alternative definition of 'paraconsistent' and, given that logical consequence is transitive, it is equivalent to our definition.[1] For this reason we call inconsistent but non-trivial theories *paraconsistent*. The equivalence indicates one reason why paraconsistent logics are worthy of study. For there are important inconsistent theories which are not trivial. Any analysis of their logical structure must therefore be done using a paraconsistent logic. Clearly, to adopt an explosive logic such as Frege/Russell or intuitionist logic would trivialise them.

History abounds with examples of paraconsistent theories: the Newton-Leibniz version of the calculus, Cantor's set theory, early quantum mechanics, Hegel's dialectic, etc. And we might add to the list, certain other

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[0] If a logic is defined in terms of a set of theses it may have more than one associated consequence relation. For example, $\{A_1 \ldots A_n\} \Vdash B$ iff $\vdash (A_1 \wedge \ldots \wedge A_n) \to B$ or $\vdash A_1 \to (\ldots \to (A_n \to B) \ldots)$ In this case all its associated consequence relations should be paraconsistent.

[1] The proof is this: if Σ is an inconsistent but non-trivial theory, then obviously the consequence relation is paraconsistent. Conversely, suppose that $\{A, \sim A\}$ *non* $\Vdash B$. Let Σ be the transitive closure of $\{A, \sim A\}$ under logical consequence. Then Σ is inconsistent but $B \notin \Sigma$.

bodies of information, which, whilst not theories in the standard sense can be thought of as logically closed. These include many bodies of law, particularly constitutions. The thesis that these theories are non-trivial but inconsistent, and not just *prima facie* inconsistent, can be given a solid philosophical basis.[2]

One paraconsistent theory has been particularly important for the motivation of paraconsistent logic: naive set theory. Since we will have several occasions to refer to it we will specify it now. Naive set theory is the theory in a first order (intensional) language whose only predicate is '$\in$', and whose postulates are

1) $\exists y \forall x\, (x \in y \leftrightarrow \varphi)$ where φ is arbitrary
2) $\{\forall x\, (x \in z \leftrightarrow x \in y)\} \Vdash z = y$

This theory captures the naive notion of set, viz. a set is the extension of an arbitrary property. Russell's paradoxes and similar contradictions are simply forthcoming by the usual arguments.

We can take paraconsistency to be the view that there are important paraconsistent theories. A position that needs to be distinguished from paraconsistency is the view that there are certain true contradictions. We will use "dialetheia" to mean "true contradiction". Thus we can call this position dialetheism. Obviously dialetheism implies paraconsistency since if there are dialetheias in some non-trivial domain, the set of true sentences of that domain will be an important paraconsistent theory. The converse implication does not hold however. One might hold that even though the Truth is consistent there are paraconsistent theories which are interesting and important, perhaps sometimes because they approximate to the truth.

Like many novel theories (including Cantor's theory of the infinite and special relativity), dialetheism runs against deeply seated views, and people tend to find it baffling. However unfamiliarity is not an argument against a view, and cogent arguments against dialetheism are much more difficult to find than philosophers have thought.

Prima facie examples of dialetheias are fairly easily produced by considering multicriterial terms, dialectical situations, etc. But perhaps the most persuasive examples are the logical paradoxes. The set theoretic and semantic paradoxes are notorious. They appear to be perfectly sound arguments with contradictory conclusions and, of course, if they are, dialetheism is true. Moreover all attempts to diagnose a failure in the arguments remain problematic, even after 80 years of their intense study.

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[2] We will not attempt this here since we have done so elsewhere. All the claims in this section are argued at greater length in *Paraconsistent Logic,* eds. G. Priest, R. Routley and J. Norman, Philosophia Verlag (forthcoming), ch. 1.

Furthermore there are theoretical reasons why any "solution" will be inadequate. These issues are discussed in the paper by Priest in this collection.

If dialetheism is correct, then Logic (the theory of the correct, most general principles of inference) must obviously be paraconsistent. However, it may well be that with the correct understanding Frege/Russell logic can be used in certain restricted domains, viz. consistent ones. This is in fact so, though obtaining a correct understanding of the matter is a sensitive business.

2. A brief history of paraconsistent symbolic logic

It is certainly possible to point to figures in the history of philosophy who at least made allowance for non-trivial inconsistent theories or worlds, or who must have accepted the idea that the correct logical consequence relation is paraconsistent. Any dialetheist, such as Hegel, must have had to do so on pain of triviality of his philosophy. However formal paraconsistent logics are a creature of this century. Their design is, in a sense, a reaction to classical (i.e. Frege/Russell) logic.

The dominant logical paradigm before this century was, of course, Aristotelian logic. The major part of this was the theory of the syllogism. Though Aristotelians held that a contradiction cannot be true, Aristotelian syllogistic is not explosive. However, like a purely positive logic it is not paraconsistent either. The point is that the poverty of the forms of syllogistic inference and its associated grammatical forms makes it impossible to ask the question of what follows from a contradiction.

However it is quite possible to build on to Aristotelean syllogistic the machinery for expressing this problem. One way, used in the C19th, is by the theory of immediate inference inherited from the Stoics. Another is by adding a new class of judgements '*S* is *P* and not-*P*' and considering rules for the non-trivial consequences of a member of this class. This latter possibility was investigated by the Russian logician Vasil'ev about 1911.[3]

The paradigm that replaced Aristotelian logic, viz. classical logic was, of course, anything but paraconsistent. The Frege/Russell account of logical consequence was the legitimate descendant of certain medieval accounts of implication. What was much more revolutionary than their logic itself was the methodology they brought to logic. The methodological techniques they used, such as a separate analysis of the quantifier, axiomatization, and, in a rudimentary form, the syntactic/semantic distinction, revolutionised our conception of what a formal logic should be like.

[3] For a full discussion of the material in this section and the rest of the history of paraconsistent logic, see Priest and others, *op cit.*, ch. 1.

The first person to conceive of the possibility of a paraconsistent formal logic, in the modern sense, was probably Łukasiewicz (1910). However, the first person to produce one was his pupil Jaśkowski (1948). Jaśkowski's basic idea is to take 'true' to be 'true according to the position of some person (e.g. in a discussion)'. This we can represent logically as 'true in some possible world (the world of that person's position)'. Then a pair of formulas A, $\sim A$, can be "true" without an arbitrary formula B being true.

In the 1950s work on paraconsistent logic began independently in South America. Asenjo (1954) and da Costa started to study paraconsistent systems. Of these the most widely developed systems are those of da Costa. His approach was basically to graft on to ordinary positive logic a "negation" operator which is not truth functional. If A takes the value 0, then $\sim A$ takes the value 1. But if A takes the value 1, $\sim A$ may have value 1 or 0.

A third, and again independent, approach can also be traced back to the late 1950s. At this time in North America Anderson and Belnap, taking off from the work of Ackermann, started to produce logical systems that were relevant i.e. which avoided the paradoxes of implication. For present purposes we can define a relevant propositional logic to be one in which if $\{A_1 \ldots A_n\} \Vdash B$, B and $A_1 \wedge \ldots \wedge A_n$ share a propositional parameter. Anderson and Belnap's intention was not to produce a paraconsistent logic as such. However their logics were paraconsistent. The paraconsistent aspect of relevant logic was later taken up in Australia by the present authors. Needless to say, we think that the relevant approach to paraconsistency is best, though we will not argue it here.[4]

Paraconsistent logic is now a rapidly growing and widely spreading subject. All three of the approaches cited above (and some others) together with their applications and philosophical *rationale* are being investigated. The main centres at the moment are Australia, Eastern Europe and South America, though there are a growing number of workers in Western Europe. It must be confessed that in North America paraconsistency has, until very recently, fallen on largely stony ground.

3. Formal paraconsistent logic

We will now discuss the three basic approaches to paraconsistency in a little more technical detail.[5] We can call the three approaches (in the order we introduced them above) the non-adjunctive approach, the "positive logic plus" approach and the relevant approach.

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[4] The argument can be found in Priest and others, *op cit.*, ch. 5.

[5] A longer discussion of the material in this section can be found in Priest and others, *loc, cit,*

a) The non-adjunctive approach

The salient feature of this approach, as the name suggests, is the rejection of adjunction: $A,\ B/A \wedge B$. This arises straightforwardly from Jaśkowski's approach as already explained, in the following way. If M is a Kripke model of some modal logic, say **S5**, let us say that A is M-true iff $\Diamond A$ is true in M. Then we define $\{A_1 \ldots A_n\} \Vdash B$ as: for all M either A_1 or $\ldots A_n$ is not M-true, or B is M-true. The failure of $\{A, \sim A\} \Vdash A \wedge \sim A$ is now apparent. But if we think of the consequence relation as relating purely truth-functional formulas, it is a fairly useless one. For it can be shown that $\{A_1 \ldots A_n\} \Vdash B$ iff for some $1 \leqslant i \leqslant n$, $\{A_i\} \Vdash B$. Moreover $\{A_i\} \Vdash B$ iff B is an ordinary two valued consequence of A_i. Thus there is no essentially multipremiss inference, and the single premiss case is just classical.

One solution to this problem is to consider various intensional connectives. Strict implication would do, but following Jaśkowski people have used a stronger "implication" called 'discursive implication', $\supset_d$. $A \supset_d B$ is defined simply as $\Diamond A \supset B$. It is easily checked that $\{A, A \supset_d B\} \Vdash B$ and Jaśkowski showed that the pure $\supset_d$ fragment of the logic is the pure calculus of material implication. Clearly however larger fragments do not coincide with their classical counterparts. For example $A \supset_d (\sim A \supset_d B)$ must fail.

This approach to paraconsistency has been generalized by a number of writers, for example da Costa and Kotas.[6] In the present collection Błaszczuk investigates the family of logics obtained by taking an arbitrary normal modal logic instead of **S5** and defining 'A is M-true' as 'ψA is true in M', where ψ is an arbitrary but fixed modality (i.e. string of $\Diamond$'s, $\Box$'s and negation signs).

Another solution to the problem of multipremiss inference is to allow, in effect, a certain amount of conjoining of premisses. But obviously we cannot conjoin them all (or we are back to classical logical consequence). So what conjoining can we do? An answer to this question has been worked out by Schotch and Jennings.[7] In essence, we are allowed to conjoin premisses up to maximal consistency. Specifically let Σ be a finite set of formulas. A *partition* of Σ of size n is a family of sets $\{\sigma_i | i \in n\}$ such that $\bigcup_{i \in n} \sigma_i = \Sigma$ and if for $i, j \in n$, $\sigma_i \cap \sigma_j \neq \emptyset$ then $i = j$. The *level* of Σ, $l(\Sigma)$ is the least n such that there is a partition of Σ of size n of which all members are (classically) consistent. If there is no such n, then, conventionally, $l(\Sigma) = \infty$. Logical consequence can now be defined as follows:

$\Sigma \Vdash B$ iff $l(\Sigma) = \infty$ or

[6] J. Kotas, and N. da Costa "On the problem of Jaśkowski and the Logics of Łukasiewicz", *Proc. of the 1st Brazilian Conference of Mathematical Logic*, A. Arruda *et al.* N. Holland, 1978.

[7] P. K. Schotch and R. E. Jennings "Inference and Necessity", *Journal of Philosophical Logic* 9, 1980, 329 - 340.

$l(\Sigma)$ is finite and for every partition of Σ of level $l(\Sigma)$ there is some member of the partition σ such that B is a two valued consequence of σ.

Obviously for consistent Σ, $\Vdash$ is the same as classical $\vdash$. However, adjunction, of course, still fails. In the paper in this collection, Schotch and Jennings consider certain generalizations of the above idea and also investigate the connection with modal semantics.

Our aim here is not to evaluate the various approaches to paraconsistency. However it will be quite clear already that non-adjunctive approaches to paraconsistency do not take the idea of a dialetheias seriously. For we have $\{A \wedge {\sim} A\} \Vdash B$, and the only thing that prevents $\{A, {\sim} A\}$ from blowing up, is the non-standard behaviour of conjunction. For this reason non-adjunctive paraconsistent logics are unsuitable as the underlying logic of important inconsistent theories such as naive set theory. For classically

$$\{\forall x(x \in R \leftrightarrow x \notin x)\} \vdash B$$

and since the non-adjunctive $\Vdash$ coincides with classical $\vdash$ in the single premiss case, the same is true of it.

Rescher and Brandom, who also pursue a non-adjunctive approach suggest[8] that instances of the abstraction scheme which lead to triviality should be considered as two non-conjoined formulas (e.g. $\forall x(x \in R \rightarrow x \notin x)$ and $\forall x(x \notin x \rightarrow x \in R)$). However, whatever the outcome of this approach, it in fact gives the game away. It concedes the crucial point, that we cannot consider naive set theory itself as an integral, coherent theory.

β) The positive logic plus approach

To be able to formalise naive set theory we need a logic which rejects the principle $\{A \wedge {\sim} A\} \Vdash B$, and if we have this, we can obviously allow adjunction with impunity. In fact, we can keep the whole of the positive logic standard but merely allow for a non-classical behaviour of negation. This brings us to the "positive logic plus approach". This starts from the assumption that positive logic (sometimes classical and sometimes intuitionistic) is correct, and adds to it a suitable negation. This may, of course, be done in different ways. One way is the way familiar from intuitionism of defining ${\sim} A$ as $A \supset f$ where f is a trivialising proposition. Obviously this will not give a paraconsistent logic. However we could take instead of f certain other formulas. In his note in this collection Bunder considers various possible candidates.

A different way of adding negation to positive logic has been pursued by da Costa and his colleagues. Essentially this amounts to taking a valuation-

[8] N. Rescher and R. Brandom, *The Logic of Inconsistency*, Blackwell, 1980. Ch. 10.

al semantics for a positive logic and then specifying the conditions on the evaluation of $\sim A$ *de novo.* The simplest example of this is provided by taking the standard two valued valuations for classical positive logic and then requiring that a valuation ν be such that

$$\text{if } \nu(A) = 0, \nu(\sim A) = 1^9 \quad (*)$$

$\Sigma \Vdash B$ can now be defined in the obvious way:

$$\Sigma \Vdash B \text{ iff for all } \nu, \text{ either } \exists A \in \Sigma \nu(A) = 0 \text{ or } \nu(B) = 1.$$

Stronger paraconsistent logics can be obtained by adding further conditions on valuations of formulas containing negation, though we cannot require the converse of $(*)$ without producing classical logic. In a similar way, if we take a suitable valuational semantics for positive intuitionist logic and add the conditions $(*)$ and

$$\text{if } \nu(\sim\sim A) = 1, \ \nu(A) = 1 \quad (**)$$

we obtain da Costa's system C_ω. The addition of further conditions produces members of da Costa's hierarchy of systems $C_n, 1 \leqslant n$.

In his paper in this collection, Alves works with the predicate extension of the system which is the same as C_1 except that $(**)$ is strengthened to a bi-conditional. He shows that many of the standard results of classical model theory have natural analogues in the model theory of this system.

It will be fairly clear from what we have said that the "negation", of da Costa's systems is fairly weak. (Indeed its semantical conditiont $(*)$, suggests that '$\sim$' is not really a contradictory forming operator a all but a subcontrary forming operator. Thus although there are true formulas of the form $A \wedge \sim A$ one might argue that the notion of a true *contradiction* is not taken seriously). Given the basic assumption of this approach this is no accident. For given the strength of the positive part of the logic, even fairly mild negation principles collapse the logic into a classical one. For example the fact that the $\boldsymbol{C}$ systems contain the paradox of implication

$$A \supset (B \supset A)$$

means that adding contraposition

$$(A \supset B) \supset (\sim B \supset \sim A)$$

almost immediately produces the unacceptable

$$A \supset (\sim A \supset B).$$

The weakness of negation makes da Costa's logics somewhat problematical. For example, the failure of contraposition results in the general failure

[9] See D. Batens "Paraconsistent Extensional Propositional Logics", *Logique et Analyse,* 1980, 90-91, 195-234.

of the principle of the substitutivity of provable equivalents:

$$\text{If} \Vdash A \equiv B \quad \text{then} \quad \Vdash C(A) \equiv C(B)$$

(where $\equiv$ is defined as usual). This in turn implies that we cannot produce a Lindenbaum algebra for the $\boldsymbol{C}$ systems in the normal way. In fact Mortensen has proved that a non-trivial Lindenbaum algebra for $\boldsymbol{C}_1$ cannot be produced in *any* way.[10] The fact that there is no Lindenbaum algebra might not seem to be a substantial philosophical (as opposed to technical) problem but in fact it is. For it implies that there are no recursive semantics of a suitable kind.[11] There are well-known arguments for the fact that philosophically adequate semantics must be recursive.

Notwithstanding the above, there can, of course, be algebraic structures which are related to the $\boldsymbol{C}$ logics in interesting ways. In their paper in this collection Carnielli and Alcantara discuss certain such structures which they call 'da Costa algebras', and for which they prove a suitable representation theorem.

γ) The relevant approach

The fact that the problems of the previous approach stem from the strength of the positive logic suggests that this should be weakened. Of course, there are independent arguments for rejecting such pure implicational formulas as $A \supset (B \supset A)$ and this brings us to the last of the three approaches to paraconsistency we mentioned: the relevant one. A consequence relation for a propositional language is relevant, if, wherever it holds, there is a propositional variable shared between the conclusion and a premiss. Relevant logics may be approached in many different ways. Even if we restrict ourselves to semantic approaches, there are still several.

The most long-standing semantics for relevant logics are those of Routley and Meyer. These are a world-type semantics. Conjunction and disjunction behave in the usual way at each world, viz. $\nu(A \wedge B \quad \omega) = 1$ iff $\nu(A \quad \omega) = 1$ and $\nu(B \quad \omega) = 1$; $\nu(A \vee B \quad \omega) = 1$ iff $\nu(A \quad \omega) = 1$ or $\nu(B \quad \omega) = 1$; but the most significant aspect of the semantics from a paraconsistent point of view is the treatment of negation. Each world ω, is correlated with an "opposite" ω^* (such that $\omega^{**} = \omega$). The truth condition for negation is then simply $\nu({\sim} A \quad \omega) = 1$ iff $\nu(A \quad \omega^*) \neq 1$.

It is a simple exercise to show that there may be a ν and a ω such that $\nu(A \wedge {\sim} A \quad \omega) = 1$. The other major aspect of the semantics is the truth condition for $\rightarrow$. The standard truth condition for $\dashv$ (strict implication)

[10] C. Mortensen "Every Quotient Algebra for $\boldsymbol{C}_1$ is Trivial " *Notre Dame Journal of Formal Logic* XXI, 1980, 694-700.

[11] See Priest and others, *op cit.*, ch. 5, fn. 29.

uses a binary relation. Those for $\rightarrow$ use a similar ternary relation and are given thus: $v(A \rightarrow B \quad \omega) = 1$ iff for all a, b such that $R\omega ab$ either $v(A \quad a) \neq 1$ or $v(B \quad b) = 1$. $\{A_1 \ldots A_n\} \Vdash B$ iff $v(A_1 \wedge \ldots \wedge A_n \rightarrow B \ \mathbf{T}) = 1$, where $\mathbf{T}$ is the base world of the model. As with modal logic, various conditions on R give rise to various logics.

It might be thought that the way a paraconsistentist ought to proceed is to let evaluations range not over $\{0, 1\}$ but over the power set of $\{0, 1\}$ (so that an evaluation of $\{0, 1\}$ corresponds to a sentence being both true and false). In his paper in this collection Routley shows how the semantics of relevant logics can be reworked in this way. If this is done, the need for the $*$ operation disappears. The cost of the reworking is that we have to add another relation S which does for 0 what R does for 1.

All logics we have considered so far have sets of theses which are subsets of that of classical logic. However it is not difficult to find plausible theses which are not theorems of classical logic. One of these is the connexivist principle, commonly called 'Aristotle', $\sim(A \rightarrow \sim A)$. When Aristotle is added to classical logic inconsistency, indeed triviality, results. This need not be the case however if Aristotle is added to a relevant logic. In his paper in this collection Mortensen discusses the semantic condition of Aristotle in a slight generalisation of Routley and Meyer semantics suitable for consistent connexive logics. If Aristotle is added to normal relevant logics, inconsistency but not triviality results. Mortensen also discusses the appropriate semantics for the Anderson and Belnap system $\boldsymbol{E}$ plus Aristotle. This is the only logic discussed in this volume which is not only paraconsistent but inconsistent.

Ex falso quodlibet is obviously a crucial principle from a paraconsistent point of view. But a principle almost as important (though not nearly so widely debated) is the assertion principle

$$\big(A \wedge (A \rightarrow B)\big) \rightarrow B.^{12}$$

The reason is that many important theories are trivialised by this principle. For example, naive set theory is trivialised thus:

(1)	$\forall x(x \in C \leftrightarrow (x \in x \rightarrow p))$	Abstraction
(2)	$C \in C \leftrightarrow (C \in C \rightarrow p)$	Instantiation from (1)
(3)	$C \in C \wedge (C \in C \rightarrow p) \rightarrow p$	Assertion
(4)	$C \in C \wedge C \in C \rightarrow p$	Substitutivity of bi-entailments from (2), (3)
(5)	$C \in C \rightarrow p$	Since $A \rightarrow A \wedge A$
(6)	$C \in C$	Modus ponens from (5), (2)
(7)	p	Modus ponens from (5), (6)

[12] Which should not be confused with modus ponens $\{A, A \rightarrow B\} \Vdash B$.

In fact, this is just a variant of the Curry—Moh Shaw Kwei paradox.[13] Derivation of the pardox usually proceeds from the absorption principle W,

$$\{A \rightarrow (A \rightarrow B)\} \Vdash (A \rightarrow B).$$

However under weak assumptions W is a simple consequence of the assertion principle.[14] Thus, any logic which admits W is not completely satisfactory as a paraconsistent logic. In fact, all the logics we considered in the first two approaches admit W, as do a number of relevant logics, including the original and elect systems of Anderson and Belnap's work, ***E***, ***R***, and ***T***. This makes it important to investigate relevant logics in which W (and consequently assertion) fail. In his paper in this collection Slaney investigates the logics obtained by dropping the principle W from ***R***, ***E***, and ***T***, using a proof technique of Meyer called metavaluation.[15]

What, however, is wrong with assertion? There are several possible answers to this. According to one, logical consequence should take notice of the number of times a premiss is *used* in inferring the conclusion. Thus since $A \wedge (A \rightarrow B)$ is used twice in inferring B — once to obtain A and once to obtain $A \rightarrow B$, $\{A \wedge (A \rightarrow B),\ A \wedge (A \rightarrow B)\} \Vdash B$ is acceptable whilst assertion is not. Of course, the set in this expression now needs to be understood as a multiset and not an ordinary one.[16]

Another possibility[17] is that we might conceive implications to form a hierarchy with the number of nested implications determining the level of a formula in the hierarchy. The suggestion then is that the levels are immiscible, in the sense that to get to a conclusion of level n we must have a premiss of level n present. This obviously rules out W. Moreover, it seems plausible to suppose that the premiss of a correct logical consequence must be relevant to the conclusion at the same level, and a natural way to express this in a propositional logic is by the demand that if $\{A\} \Vdash B$, A and B must have a common variable at the same "depth" (i.e. to the same degree of nesting within $\rightarrow$'s). This condition has been called by Brady, for obvious reasons, "depth relevance". Depth relevance thus provides a necessary condition for a correct logical consequence and the antecedent of the assertion principle is not depth relevant to its consequent. In his paper in this collection Brady proves the depth relevance of an important class of relevant logics.

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[13] See R. Meyer, R. Routley and J. Dunn "Curry's Paradox", *Analysis* 39, 1979, 124 - 8.

[14] See § 3.9, R. Routley *et al.*, *Relevant Logics and Their Rivals*, Ridgeview, California, 1982.

[15] See A. Anderson and N. Belnap, *Entailment*, Princeton U.P., 1975, § 22.3.

[16] This line of thought is taken further in R. Meyer and M. McRobbie 'Multisets and Relevant Implication', *Australasian Journal of Philosophy*, 60, 1982, 107-139.

[17] Suggested by J. Myhill "Levels of Implication" in *The Logical Enterprise* eds. A. Anderson *et al.*, Yale U.P., 1975.

A somewhat different approach, which results in a relevant logic is that of Tennant in this collection. Consider the sequent corresponding to *ex falso quodlibet* $A \wedge \sim A \ :B$. This is classically unfalsifiable. However it owes this fact to the classical unverifiability of its antecedent. This shows that it is $A \wedge \sim A$: which is the basic logical fact. The B is just "noise". Let us call a sequent *perfectly valid* if it is classically valid but has no classically valid proper subsequent. What we are interested in then is sequents that are perfectly valid, or rather, since a logic should be closed under substitution, sequents that can be obtained from perfectly valid sequents by substitution. Tennant calls these *Entailments*. Entailment is easily seen to be relevant.

Another fact about Entailments, easily checked, is that if B is a classical logical consequence of some consistent set Σ, then some finite subset of Σ Entails B. Thus Entailment preserves the strength of classical logic as far as the deduction of theorems from consistent axioms goes. However, transitivity fails for Entailment. Each of the following is an Entailment.

$$p \wedge \sim p \ :p \wedge (\sim p \vee q), \qquad p \wedge (\sim p \vee q) \ :q,$$

whilst $p \wedge \sim p \ :q$ is not.

Still it is possible to characterise those situations in which transitivity holds in a very simple and natural way.

The second of the above Entailments is, of course, the disjunctive syllogism. Standard relevant logics admit transitivity but not the disjunctive syllogism. The disjunctive syllogism is just the assertion principle for material implication and, in virtue of what we have said about assertion, it is not surprising that Entailment, satisfying the disjunctive syllogism, permits Curry-type paradoxes (though in virtue of the general failure of transitivity it does not follow immediately). Thus Entailment is unsuitable as the relation of logical consequences for many inconsistent theories.

4. Inconsistent theories

Having surveyed the various approaches to paraconsistent logic, let us move on to their use in formalising inconsistent theories.[18] One of the most significant advances in 19th century mathematics was Cantor's investigation of the infinite. Until then the infinite had often been thought of as amorphous and sometimes beyond the bounds of rational investigation. Cantor showed that the infinite has a determinate and important structure investigable by quite rational techniques. In many ways, what Cantor did for the infinite, paraconsistency does for the inconsistent. That the inconsistent has a determinate structure discoverable by rational

[18] The material in this section is discussed in greater detail in Priest and others, *op cit.*, ch. 14.

investigation is no longer in doubt. What that is, is still largely an open question.

This will be determined only by the investigation of inconsistent theories, which investigations are still in their infancies. The only theory that has received any work so far is naive set theory. Inconsistent sets have been shown to have some interesting but not very surprising properties.[19] Perhaps the most surprising result so far is the proof of the axiom of choice in naive set theory.[20] This is well known to be independent of virtually all standard set theories.

There are many other inconsistent theories which beg to be investigated. One is naive semantics, the theory of semantically closed languages (see the article by Priest in this collection). Two more are worth a special but brief mention. One is the early theory of the calculus, particularly the theory of infinitesimals. At different times in the calculation of a derivative it has to be assumed that an infinitesimal is non-zero and that it is zero. This was pointed out by many contemporary writers, such as Berkeley. (It is sometimes suggested that Robinson's reworking of the calculus in non-standard analysis shows that the early calculus was really consistent. But though nonstandard analysis is a beautiful theory, to suppose that it captures the Leibnizian infinitesimal calculus is to commit a gross anachronism.)

The other theory worth a special mention is quantum theory. At numerous points this verges on, or into, the inconsistent (for example, in the area of the Dirac δ function[21]). Of particular interest is also the use of paraconsistency in connection with casual anomalies. Quantum logic avoids the inconsistency generated by the two-slit experiment by dropping distributivity. A paraconsistent logic could allow the contradiction to be derived: because of the failure of *ex falso quodlibet* this would not matter. But what consequences (even empirical consequences) this would have has scarcely been investigated.[22]

So much for what is provable about the inconsistent. The other side of this question is of course, 'What isn't provable?'. The only theory that has been investigated in this respect is naive set theory and again, investigations are in their infancy. The question of the underlying logic is here absolutely crucial. Arruda and da Costa in their article in this volume, discuss naive set theory based on a weak relevant system ***P***. This theory

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[19] See N. C. Costa "On the Theory of Inconsistent Formal Systems", *Notre Dame Journal of Formal Logic* XV, 1974, 497-510.

[20] See R. Routley "Ultralogic as Universal" printed as the Appendix in *Exploring Meinong's Jungle and Beyond*, A. N. U., 1980.

[21] We owe this point to C. Mortensen.

[22] But see Routley *op cit.*

is certainly inconsistent, but they show that there are formulas which are not provably entailed by the axioms of the logic and set theory. The strongest result in this area is due to Brady[23] who has proved that set theory based on quite a strong depth relevant logic is inconsistent but non-trivial. It is known that set theory based on a slightly stronger relevant (but depth irrelevant) logic is not only inconsistent but trivial.[24] Of course, proving that *some* things are not provable is only the first step. Determining *what* isn't provable in various of these theories is the next.

The most celebrated result about what is not provable in theories based on classical logic is, of course, Gödel's incompleteness theorem. Since the assumption of consistency is an important part of proofs of this theorem, the standard proof techniques cannot be applied to inconsistent theories. Thus the questions of the completness of naive set theory, semantically closed arithmetic, etc., are open. It is worth noting that Gödel's second incompleteness theorem (concerning the unprovability of consistency) fails even for consistent theories based on relevant logics.[25]

5. Concluding remarks: wider horizons

In this final section we will indicate just a couple of the wider aspects of paraconsistency. First, it will already be clear that there are many logical problems in the field of paraconsistency that require work. However there are several more which we have not indicated. There is, for example, a host of problems concerning quantification and relevant logic.[26] The area of paraconsistent tense and deontic logics has hardly been touched[27] and neither has the connection between paraconsistency and vagueness.[28] These are just three of the more open areas.

Second, it is evident that paraconsistency and particularly dialetheism have deep philosophical consequences. Much traditional as well as common-sense thought is predicted on the assumption that contradictions are uniformly false/unacceptable/disastrous. This assumption ***must be shaken off***, or at least restricted to its legitimate domain: the consistent. Here again, the comparison with infinity is useful. Many of our conceptions

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[23] R. Brady "The Non-Triviality of Dialectical Set Theory" in Priest and others, *op cit.*

[24] See J. Slaney "*RWX* is not Curry Paraconsistent" in Priest and others, *op cit.*

[25] See R. Meyer "Relevant Arithmetic", ***Polish Academy of Science, Bulletin of the Section of Logic*** 5(4), 1976.

[26] See R. Routley "Problems and Solutions in the Semantics of Quantified Relevant Logics", ***Mathematical Logic in Latin America,*** N. Holland, 1980. eds. Arruda, ***et al.***

[27] But see G. Priest "To be and not to be: Dialectial Tense Logic", ***Studia Logica*** 41, Nos. 2/3, 1982, 249-268.

[28] But see C. Pinter "Logic of Inherent Ambiguity", ***Proc. of the 3rd Brazilian Conference on Mathematical Logic,*** eds. A. Arruda *et al.*, N. Holland, 1980.

concerning size, such as that the whole must be greater than the part, are drawn from the finite. The realization that these principles do not apply universally but only in finite domains was an important one. The realisation that, say, disjunctive syllogism is applicable only in consistent domains is, we think, similarly important.

This is not the place to trace the philosophical implications of paraconsistency.[29] However one that is hard to avoid mentioning is the effect of paraconsistency on dialectics. The notion of a logical contradiction has always been central to dialectics.[30] And because of a widespread belief that contradictions cannot be realised, dialectics has often been written off as incoherent (especially in Anglo-American philosophy). Alternatively, the notion of contradiction has been weakened (to e.g. that of opposing tendencies), thus distorting the dialectical tradition. In putting an end to these reactionary and revisionary tendencies, paraconsistency will have a liberating effect on the study of dialectics. In his article in this volume Smolenov discusses one paraconsistent approach to dialectics.

However, consistency assumptions are not so easy to shake off, precisely because reasoning is so commonly predicated on them. For example, it is often said even by the exponents of paraconsistent logic[31] that a paraconsistent logic should not contain the law of non-contradiction $\sim(A \wedge \sim A)$. Why not? Presumably because in an inconsistent theory we will have theorems of the form $A \wedge \sim A$. But of course this argument succeeds only if a consistency assumption is made! In this respect even paraconsistent logicians are prone to lapse into the consistency habit. However they have at least emancipated themselves from consistency, and no longer live in superstitious fear and awe of contradictions.[32]

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[29] Some of these are discussed in Priest and others, *op cit.*, ch. 19.

[30] See Priest and others, *op cit.*, ch. 2.

[31] See for example da Costa *op cit.*

[32] L. Wittgenstein, *Philosophical Remarks*, Blackwell, 1964, p. 332 and *Remarks on the Foundations of Mathematics*, Blackwell, 1956, p. 53.